Exercise 27

Suppose the system below is consistent for all possible values of f and g. What can you say about the coefficients c and d? Justify your answer.

$$\begin{aligned} x_1 + 3x_2 &= f \\ cx_1 + dx_2 &= g \end{aligned}$$

Solution

Write the augmented matrix corresponding to this system of equations.

$$\left[\begin{array}{cc|c} 1 & 3 & f \\ c & d & g \end{array}\right]$$

To make the bottom left entry 0, multiply the first row by $-c$ and add it to the second row.

$$\left[\begin{array}{cc|c} 1 & 3 & f \\ 0 & d-3c & g-cf \end{array}\right] \tag{1}$$

Divide the second row by $d - 3c$, assuming it's not zero.

$$\left[\begin{array}{cc|c} 1 & 3 & f \\ 0 & 1 & \frac{g-cf}{d-3c} \end{array}\right]$$

Multiply the second row by -3 and add it to the first row.

$$\left[\begin{array}{cc|c} 1 & 0 & f-3\frac{g-cf}{d-3c} \\ 0 & 1 & \frac{g-cf}{d-3c} \end{array}\right]$$

Simplify the matrix.

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{df-3g}{d-3c} \\ 0 & 1 & \frac{g-cf}{d-3c} \end{array}\right]$$

The solution is then

$$x_1 = \frac{df-3g}{d-3c} \quad \text{and} \quad x_2 = \frac{g-cf}{d-3c}$$

if $d - 3c \neq 0$. If $d - 3c = 0$, then equation (1) becomes

$$\left[\begin{array}{cc|c} 1 & 3 & f \\ 0 & 0 & g-cf \end{array}\right].$$

The last row implies that $0x_1 + 0x_2 = g - cf$. No values of x_1 and x_2 can make this a true statement, since the right side is nonzero. Therefore, there's no solution if $d - 3c = 0$.